

THE BRIANÇON-SKODA THEOREM AND COEFFICIENT IDEALS FOR NON \mathfrak{m} -PRIMARY IDEALS

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ABSTRACT. We generalize a Briançon-Skoda type theorem first studied by Aberbach and Huneke. With some conditions on a regular local ring (R, \mathfrak{m}) containing a field, and an ideal I of R with analytic spread ℓ and a minimal reduction J , we prove that for all $w \geq -1$, $\overline{I^{\ell+w}} \subseteq J^{w+1} \mathfrak{a}(I, J)$, where $\mathfrak{a}(I, J)$ is the coefficient ideal of I relative to J , i.e. the largest ideal \mathfrak{b} such that $I\mathfrak{b} = J\mathfrak{b}$. Previously, this result was known only for \mathfrak{m} -primary ideals.

1. INTRODUCTION

Throughout this paper all rings are assumed to be commutative, Noetherian and with identity.

The classical Briançon-Skoda theorem, proved first by Briançon and Skoda in the complex analytic case [5], and by Lipman and Sathaye for regular rings in general [8], states that if (R, \mathfrak{m}) is a regular local ring, then given an ideal I of analytic spread ℓ , and a reduction J of I , we have $\overline{I^{\ell+w}} \subseteq J^{w+1}$ for $w \geq 0$. Further refinements of this theorem have abounded. One such refinement is (see Section 2 for the definition of the coefficient ideal $\mathfrak{a}(I, J)$):

Theorem 1.1. ([3], Theorem 2.7) *Let (R, \mathfrak{m}) be a regular local ring of dimension d containing a field and having an infinite residue field. Let I be an \mathfrak{m} -primary ideal and let J be a minimal reduction of I . Then for all $w \geq -1$,*

$$\overline{I^{d+w}} \subseteq J^{w+1} \mathfrak{a}(I, J).$$

Note that this theorem applies only to \mathfrak{m} -primary ideals I . The reason is that the proof relies on an iteration giving a descending sequence of ideals, all of which contain a fixed power of I . Thus, in the \mathfrak{m} -primary case, this descending sequence stabilizes, and the stable value is shown to be the desired value. Therefore, the same proof will not work in the non- \mathfrak{m} -primary case. The main result of this paper (see Theorem 3.4) extends Theorem 1.1 to regular rings where a certain quotient (depending on I) is complete—in particular, we show that the theorem is true for all ideals when R itself is complete.

There have been a number of results of this type. Some of them are in [1],[2],[3],[4],[7],[9],[11]. In particular, with the development of the theory of *tight closure* by Hochster and Huneke, these authors proved a generalized Briançon-Skoda theorem from which the original Briançon-Skoda theorem could follow. We discuss this for rings containing a field in the next section, after the definition of tight closure.

2. INTEGRAL CLOSURE, TIGHT CLOSURE AND THEOREMS OF BRIANÇON-SKODA TYPE

Recall that an element x of R is *integral* over an ideal I of R if there exists a positive integer k such that $x^k + a_1x^{k-1} + \cdots + a_k = 0$ where $a_i \in I^i$ for $1 \leq i \leq k$. The set of all elements of R that are integral over I is an ideal of R called the *integral closure* of I .

Another definition is the one of a *reduction* of an ideal that was first introduced by Northcott and Rees [10]. An ideal $J \subseteq I$ is a *reduction* of I if there exists an integer r such that $JI^r = I^{r+1}$. The least such integer is the *reduction number* of I with respect to J . A reduction J of I is called a *minimal reduction* if J is minimal with respect to inclusion among reductions. When the ring (R, m) is local with infinite residue field, every minimal reduction J of I has the same number of minimal generators. This number is called the *analytic spread* of I , denoted by $\ell(I)$, and we always have that $\text{ht}(I) \leq \ell(I) \leq \dim R$. If an ideal $J \subseteq I$ is a reduction, then $\overline{J} = \overline{I}$.

Let R be a Noetherian ring of prime characteristic $p > 0$ and let q be a varying power of p . Let R^o be the complement of the union of the minimal primes of R and let I be an ideal of R . Define $I^{[q]} = (i^q : i \in I)$, the ideal generated by the q^{th} powers of all the elements of I . The *tight closure* of I is the ideal $I^* = \{x \in R; \text{for some } c \in R^o, cx^q \in I^{[q]}, \text{ for } q \gg 0\}$. We always have that $I \subseteq I^* \subseteq \overline{I}$. If $I^* = I$ then the ideal I is said to be *tightly closed*. A ring in which every ideal is tightly closed is called *weakly F-regular*. We say that elements x_1, \dots, x_n of R are *parameters* if the height of the ideal generated by them is at least n (we allow this ideal to be the whole ring, in which case the height is said to be ∞). The ring R is said to be *F-rational* if the ideals generated by parameters are tightly closed.

The theory of tight closure gives another proof of the Briançon-Skoda theorem in characteristic p .

Theorem 2.1. ([7], Theorem 5.4) *Let R be a Noetherian ring of characteristic p , and let I be an ideal of positive height generated by n elements. Then for every $w \in \mathbb{N}$, $\overline{I^{n+w}} \subseteq (I^{w+1})^*$. In particular, $\overline{I^n} \subseteq I^*$.*

If R is weakly F-regular (in particular, if R is regular), of characteristic p , then $\overline{I^{n+w}} \subseteq I^{w+1}$ and $\overline{I^n} \subseteq I$.

It should be noted that the characteristic zero case of the original Briançon-Skoda theorem can be reduced to the characteristic p case, but tight closure does not seem to offer such a generalization for rings of mixed characteristic.

Another theorem, established by Aberbach and Huneke, allows us to replace the assumption weakly F-regular by F-rational, in the second part of Theorem 2.1. It states the following:

Theorem 2.2. ([1], Theorem 3.6) *Let (R, \mathfrak{m}) be an F-rational local ring of characteristic p , and let $I \subseteq R$ be an ideal generated by ℓ elements. Then $\overline{I^{\ell+w}} \subseteq I^{w+1}$ for all $w \geq 0$.*

If in Hochster and Huneke's Theorem 2.1 above, one replaces I by a minimal reduction J , generated by ℓ elements (assuming that the ring R is local with infinite residue field), one obtains that $\overline{I^{\ell+w}} \subseteq (J^{w+1})^*$. The relatively simple argument that is used leads one to examine the coefficients of the elements of J . For simplicity, consider the case $w = 0$. Given $z \in \overline{I^\ell} = \overline{J^\ell}$, there exists an element $c \in R^o$ such that $cz^q \in (J^\ell)^q$. Since J is generated by ℓ elements, then $cz^q \in J^{[q]}J^{(\ell-1)q}$. Further information can be obtained from taking into consideration the factor $J^{(\ell-1)q}$ and has led to results of the form $\overline{I^{\ell+w}} \subseteq J^{w+1}K$ where I is an ideal of analytic spread ℓ in a regular local ring R , J is a minimal reduction of I , and K is an ideal of coefficients.

Towards the above goal, Aberbach and Huneke introduced the following definition in [3]:

Definition 2.3. Let R be a commutative Noetherian ring and let $J \subseteq I$ be two ideals of R . The *coefficient ideal* of I relative to J , denoted by $\mathfrak{a}(I, J)$, is the largest ideal \mathfrak{b} of R for which $I\mathfrak{b} = J\mathfrak{b}$.

They were then able to prove Theorem 1.1. In the next section, we state and prove a generalization of this theorem to ideals which are not necessarily \mathfrak{m} -primary. See Theorem 3.4 for a specific statement.

3. A BRIANÇON-SKODA THEOREM WITH COEFFICIENTS

We are now ready to present the argument needed to generalize Theorem 1.1.

Notation. If $J \subseteq I$ are two ideals of R , x_1, \dots, x_n are elements of R and t is any positive integer, then \mathfrak{a}_t will denote the coefficient ideal of the ideal $I + (x_1^t, \dots, x_n^t)$ relative to the ideal $J + (x_1^t, \dots, x_n^t)$.

Lemma 3.1. $(\mathfrak{a}_t)_t$ is a decreasing sequence of ideals.

Proof. In order to prove the inclusion $\mathfrak{a}_{t+1} \subseteq \mathfrak{a}_t$, it is enough to show that the inclusion $\mathfrak{a}_{t+1}(I + (x_1^t, \dots, x_n^t)) \subseteq \mathfrak{a}_{t+1}(J + (x_1^t, \dots, x_n^t))$ holds, since this then implies that $\mathfrak{a}_{t+1}(I + (x_1^t, \dots, x_n^t)) = \mathfrak{a}_{t+1}(J + (x_1^t, \dots, x_n^t))$. But \mathfrak{a}_t is the largest ideal for which this equality holds.

Now the inclusion $\mathfrak{a}_{t+1}(I + (x_1^t, \dots, x_n^t)) \subseteq \mathfrak{a}_{t+1}(J + (x_1^t, \dots, x_n^t))$ is easy to prove since on one hand we have $\mathfrak{a}_{t+1}I \subseteq \mathfrak{a}_{t+1}(I + (x_1^{t+1}, \dots, x_n^{t+1})) = \mathfrak{a}_{t+1}(J + (x_1^{t+1}, \dots, x_n^{t+1})) \subseteq \mathfrak{a}_{t+1}(J + (x_1^t, \dots, x_n^t))$, and on the other hand, $\mathfrak{a}_{t+1}x_i^t \subseteq \mathfrak{a}_{t+1}(J + (x_i^t)) \subseteq \mathfrak{a}_{t+1}(J + (x_1^t, \dots, x_n^t))$ for all $i = 1, \dots, n$.

Hence $\mathfrak{a}_{t+1}(I + (x_1^t, \dots, x_n^t)) \subseteq \mathfrak{a}_{t+1}(J + (x_1^t, \dots, x_n^t))$. \square

Lemma 3.2. Let $\mathfrak{a} = \mathfrak{a}(I, J)$ and $\mathfrak{b} = \bigcap_t \mathfrak{a}_t$. Then $\mathfrak{a} \subseteq \mathfrak{b}$.

Proof. To prove $\mathfrak{a} \subseteq \mathfrak{a}_t$ for all t , we show that $\mathfrak{a}(I + (x_1^t, \dots, x_n^t)) \subseteq \mathfrak{a}(J + (x_1^t, \dots, x_n^t))$.

But we have that $\mathfrak{a}I = \mathfrak{a}J \subseteq \mathfrak{a}(J + (x_1^t, \dots, x_n^t))$, and also that for any $i = 1, \dots, n$, $\mathfrak{a}x_i^t \subseteq \mathfrak{a}(J + (x_i^t)) \subseteq \mathfrak{a}(J + (x_1^t, \dots, x_n^t))$.

Hence the inclusion $\mathfrak{a}(I + (x_1^t, \dots, x_n^t)) \subseteq \mathfrak{a}(J + (x_1^t, \dots, x_n^t))$ is clear. \square

We will need Chevalley's theorem in order to prove Theorem 3.4.

Theorem 3.3. ([6], Lemma 7) Let (R, \mathfrak{m}) be a complete local ring and let $\{J_n\}_n$ be a decreasing sequence of ideals with $\bigcap_n J_n = 0$. Then, for all $n \geq 1$, there exists $t_n \geq 1$, such that $J_{t_n} \subseteq \mathfrak{m}^n$.

We now present the main theorem in this paper.

Theorem 3.4. Let (R, \mathfrak{m}) be a regular local ring of dimension d containing a field. Let I be an ideal of R of analytic spread ℓ , and let J be a reduction of I . Choose x_1, \dots, x_n in R such that the ideal $I + (x_1, \dots, x_n)$ is \mathfrak{m} -primary. Let $\mathfrak{b} = \bigcap_t \mathfrak{a}_t$ (with \mathfrak{a}_t being as in the notation above), and assume that R/\mathfrak{b} is complete (in particular R itself may be complete). Then $\mathfrak{b} = \mathfrak{a}(I, J)$ and for all $w \geq -1$ we have

$$\overline{I^{\ell+w}} \subseteq J^{w+1}\mathfrak{a}(I, J).$$

Proof. Since J is a reduction of I , there exists r such that $JJ^r = I^{r+1}$ and this implies that for any ideal L of R , $(J + L)(I + L)^r = (I + L)^{r+1}$. In fact we have:

$$\begin{aligned} (I + L)^{r+1} &= I^{r+1} + I^r L + \dots + L^r I + L^{r+1} = JJ^r + L(I^r + \dots + L^{r-1}I + L^r) \\ &\subseteq (J + L)(I + L)^r \subseteq (I + L)^{r+1}. \end{aligned}$$

In particular, for all t , we have $(J + (x_1^t, \dots, x_n^t))(I + (x_1^t, \dots, x_n^t))^r = (I + (x_1^t, \dots, x_n^t))^{r+1}$. Hence for all t , $J + (x_1^t, \dots, x_n^t)$ is a reduction of $I + (x_1^t, \dots, x_n^t)$. Now apply Theorem 1.1 to the m -primary ideal $I + (x_1^t, \dots, x_n^t)$ to conclude that

$$\overline{I^{\ell+w}} \subseteq \overline{(I + (x_1^t, \dots, x_n^t))^{\ell+w}} \subseteq (J + (x_1^t, \dots, x_n^t))^{w+1} \mathfrak{a}_t.$$

Next, we show that $\mathfrak{a} = \mathfrak{b}$ where $\mathfrak{a} = \mathfrak{a}(I, J)$ is the coefficient ideal of I relative to J . We already know from Lemma 3.2 that $\mathfrak{a} \subseteq \mathfrak{b}$. If \mathfrak{b} is strictly larger than \mathfrak{a} , then $\mathfrak{b}J \neq \mathfrak{b}I$. Thus there are elements $y \in \mathfrak{b}$ and $c \in I$ with $yc \notin \mathfrak{b}J$.

We are going to prove that $yc \in \mathfrak{b}J$, and therefore by contradiction we conclude that $\mathfrak{b} = \mathfrak{a}$.

For any t , y is an element of \mathfrak{a}_t and this implies that $yc \in \mathfrak{a}_t I \subseteq \mathfrak{a}_t(I + (x_1^t, \dots, x_n^t)) = \mathfrak{a}_t(J + (x_1^t, \dots, x_n^t)) \subseteq \mathfrak{a}_t J + (x_1^t, \dots, x_n^t)$. Hence, $yc \in \bigcap_t (\mathfrak{a}_t J + (x_1^t, \dots, x_n^t)) \subseteq \bigcap_t (\mathfrak{a}_t J + m^t)$.

Since R/\mathfrak{b} is complete and (\mathfrak{a}_t) is a decreasing sequence with $\bigcap_t \mathfrak{a}_t = \mathfrak{b}$, Chevalley's theorem shows that for all $j \in \mathbb{N}$, there exists t_j such that $\mathfrak{a}_{t_j} \subseteq \mathfrak{b} + m^j$ and the sequence (t_j) can be chosen increasing. Consequently, we deduce that for any $t \geq t_1$, there exists $j_t \in \mathbb{N}$ with $\mathfrak{a}_t \subseteq \mathfrak{b} + m^{j_t}$, and such that the sequence (j_t) is increasing to infinity. This can be done by taking $j_t = k$ for all $t_k \leq t < t_{k+1}$, $k \geq 1$.

Hence if $t \geq t_1$, we obtain that $\mathfrak{a}_t J + m^t \subseteq (\mathfrak{b} + m^{j_t})J + m^t \subseteq \mathfrak{b}J + m^\lambda$ where $\lambda = \min\{j_t, t\}$. Note that λ is going to infinity as t goes to infinity. Therefore, $\bigcap_t (\mathfrak{a}_t J + m^t) \subseteq \bigcap_t (\mathfrak{b}J + m^\lambda) \subseteq \mathfrak{b}J$, by the Krull intersection theorem.

$\lambda \rightarrow \infty$ Thus we have proved that $yc \in \mathfrak{b}J$, a contradiction. The desired conclusion $\mathfrak{b} = \mathfrak{a}$ now follows.

To finish the proof of the theorem, recall that we have already proved that for all t , $\overline{I^{\ell+w}} \subseteq (J + (x_1^t, \dots, x_n^t))^{w+1} \mathfrak{a}_t$.

But for $t \gg 0$, there exists j_t such that $\mathfrak{a}_t \subseteq \mathfrak{b} + m^{j_t} = \mathfrak{a} + m^{j_t}$ and (j_t) is increasing to infinity. Hence,

$$\begin{aligned} \overline{I^{\ell+w}} &\subseteq (J + (x_1^t, \dots, x_n^t))^{w+1} \mathfrak{a}_t \\ &\subseteq J^{w+1} \mathfrak{a}_t + (x_1^t, \dots, x_n^t) \\ &\subseteq J^{w+1} (\mathfrak{a} + m^{j_t}) + m^t \\ &\subseteq J^{w+1} \mathfrak{a} + m^{\min\{j_t, t\}} \end{aligned}$$

where $\min\{j_t, t\} \rightarrow \infty$ as $t \rightarrow \infty$. By another application of the Krull intersection theorem we finally conclude that $\overline{I^{\ell+w}} \subseteq J^{w+1} \mathfrak{a}$, proving the theorem. \square

Question 3.5. Can we prove Theorem 3.4 without assuming that R/\mathfrak{b} is complete? We will have an affirmative answer if the coefficient ideal commutes with completion, i.e. if $\mathfrak{a}(I, J)\hat{R} = \mathfrak{a}(I\hat{R}, J\hat{R})$. Because if this is true, then as \hat{R} is faithfully flat, one deduces that

$$\begin{aligned} \overline{I^{\ell+w}} &= \overline{I^{\ell+w}} \hat{R} \cap R \subseteq \overline{I^{\ell+w} \hat{R}} \cap R \\ &\subseteq J^{w+1} \mathfrak{a}(I\hat{R}, J\hat{R}) \hat{R} \cap R \\ &= J^{w+1} \mathfrak{a}(I, J) \hat{R} \cap R \\ &= J^{w+1} \mathfrak{a}(I, J). \end{aligned}$$

Note that we always have $\mathfrak{a}(I, J)\hat{R} \subseteq \mathfrak{a}(I\hat{R}, J\hat{R})$. We would like to know whether the second inclusion holds in general.

Remark 3.6. We observe that the coefficient ideal does not commute with localization. Consider $J \subseteq I$ with $J_P = I_P$ for some prime P , but not equal up to integral closure. Replace J by $m^n J$. Then for $n \gg 0$, $\overline{m^n J} \subset \overline{I}$ but are not equal. Thus $\mathfrak{a}(m^n J, I) = 0$ but $\mathfrak{a}((m^n J)_P, I_P) = \mathfrak{a}(I_P, I_P) = R_P$.

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